

# Borel resummation of transverse momentum distributions

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## Abstract

We present a new prescription for the resummation of contributions due to soft gluon emission to the transverse momentum distribution of processes such as Drell-Yan production in hadronic collisions. We show that familiar difficulties in obtaining resummed results as a function of transverse momentum starting from impact-parameter space resummation are related to the divergence of the perturbative expansion of the momentum-space result. We construct a resummed expression by Borel resummation of this divergent series, removing the divergence in the Borel inversion through the inclusion of a suitable higher twist term. The ensuing resummation prescription is free of numerical instabilities, is stable upon the inclusion of subleading terms, and the original divergent perturbative series is asymptotic to it. We compare our results to those obtained using alternative prescriptions, and discuss the ambiguities related to the resummation procedure.

# 1 Transverse momentum resummation

The computation of transverse momentum distributions of heavy systems (such as dileptons, vectors bosons, Higgs) plays an important role in collider phenomenology, from the Tevatron to the LHC [1, 2]. As is well known, the perturbative QCD expansion of the inclusive distribution contains to all orders powers of  $\alpha_s \ln^2(q_T/Q)$ , due to the emission of soft and collinear gluons. When the transverse momentum  $q_T$  is much smaller than the mass of the final state  $Q$  these logs become large and must be resummed in order for perturbative predictions to remain reliable.

The resummation, to given logarithmic accuracy, can be performed [3] for the Fourier transform of the differential cross-section  $\frac{d\sigma}{dq_T^2}$  with respect to  $q_T$ . Upon Fourier transformation,  $q_T$  turns into its Fourier conjugate, the impact parameter  $b$ , and large logs of  $q_T/Q$  become large logs of  $bQ$ . Fourier transformation is necessary in order for the contributions included by resummation to respect transverse momentum conservation, thereby avoiding the spurious factorial growth of resummed coefficients [4]. However, the Fourier transform must be inverted in order to obtain resummed predictions for physical observables. This is problematic because the Fourier inversion integral necessarily involves an integration over the region of impact parameters where the strong coupling is not well defined because of the Landau pole.

This problem has been treated with various prescriptions. One possibility is to modify the behaviour of the strong coupling in the infrared in the Fourier inversion integral [3] ( $b_*$  prescription, henceforth): this procedure is widely used, but it is known to lead to numerical instabilities when the resummed results are matched to fixed-order ones [5]. A second option is based on the observation that the Fourier inversion integral can be computed order by order in an expansion of the resummed results in powers of  $\alpha_s$ : if only leading log terms are retained in the Fourier inversion, the result is then well defined for all values of  $q_T$  [5]. This procedure however is unstable to the inclusion of subleading corrections: the Fourier inversion can be performed to next-to-leading log accuracy [6] (as it is necessary if the resummation is performed to this order), but in such case the result differs significantly from the leading log one, and in fact for  $Q$  around 100 GeV it blows up for values of  $q_T$  of order of several GeV, well within the perturbative region. A “minimal” prescription which is free of these difficulties can be constructed [7], along the lines of the similar prescription for threshold resummation [4]. Namely, the integration path in the Fourier inversion is deformed in such a way as to leave unchanged the result to any finite perturbative order, but avoiding the Landau pole and associate cut in the resummed result. This leads to a prescription which is free of numerical and perturbative instabilities: its only shortcoming is that it is difficult to assess the ambiguities related to the resummation procedure, as it can be done in the  $b_*$  prescription by varying the way in which the infrared behaviour of the strong coupling is modified.

Here we shall show that, analogously to what happens in the case of threshold resummation [8], the ambiguity in the resummation procedure is due to the fact that the perturbative expansion of the resummed result for the transverse momentum distribution itself in powers of  $\alpha_s$  diverges. After discussing, in the next section of this paper, how existing prescriptions treat this divergence, we will show in section 3 that the divergent series can be treated by Borel summation, as is the case for threshold resummation [8, 9]. The Borel transform of the series converges and can be summed. The inversion integral which gives back the original series di-

verges, but the divergence can be removed by including a suitable higher twist term. This leads to a resummed result of which the original divergent series is an asymptotic expansion. The ensuing prescription is given in terms of a contour integral which is easily amenable to numerical implementation. The result is free of numerical instabilities, and stable upon the inclusion of subleading corrections. An estimate of the ambiguity on the resummed results may be obtained from a variation of the higher-twist term which is included in order to render the results convergent. In section 4 we will compare the result of our prescription to other existing prescriptions in the case of the Drell-Yan process, and discuss the ambiguities related to the resummation procedure. Some results on Fourier transforms are collected in the Appendix.

## 2 The need for a resummation prescription

Let us consider a parton-level quantity  $\bar{\Sigma}$  which depends on a large scale  $Q$  and a transverse momentum  $\vec{q}_T$ , such as the partonic Drell-Yan differential cross-section  $\frac{d\sigma}{dq_T^2}$ . Resummation is necessary because the perturbative coefficient of order  $n$  in the expansion of  $\bar{\Sigma}$  in powers of  $\alpha_s(Q^2)$  has the form

$$\bar{\Sigma} = \sum_n \alpha_s^n(Q^2) \bar{\Sigma}^{(n)}(q_T^2, Q^2) \quad (2.1)$$

$$\bar{\Sigma}^{(n)}(q_T^2, Q^2) = \left[ \frac{P_n(\ln \hat{q}_T^2)}{\hat{q}_T^2} \right]_+ + Q_n(\hat{q}_T^2) + D_n \delta(\hat{q}_T^2), \quad (2.2)$$

where

$$\hat{q}_T^2 \equiv \frac{q_T^2}{Q^2}, \quad (2.3)$$

$P_n(\ln \hat{q}_T^2)$  is a polynomial of degree  $2n - 1$  in  $\ln \hat{q}_T^2$ ,  $Q_n(\hat{q}_T^2)$  is regular as  $q_T \rightarrow 0$ , and  $D_n$  are constants (see the Appendix for a definition of the  $+$  distribution). Physical observables are obtained, exploiting collinear factorization, as the convolution of parton level cross-sections with parton distributions [3]. When  $Q^2$  is large enough, it sets the scale of parton distributions, and the  $q_T$  dependence is entirely given by the partonic cross-section. For lower values of  $Q^2$  the scale of parton distributions is set by the impact parameter  $b$ , which is Fourier conjugate to  $q_T$ , the convolution must be performed in  $b$  space, and the Fourier transform must be inverted to obtain physical predictions. In either case, the resummation is performed in  $b$  space at the level of partonic observables.

Upon Fourier transformation,  $\vec{q}_T$  is replaced by its Fourier-conjugate variable, the impact parameter  $\vec{b}$ , and the small- $q_T$  region is mapped onto the large- $b$  region. Large logs of  $b$  can then be resummed, leading to an expression of the form

$$\Sigma(\alpha_s, \bar{\alpha}L) = \sum_{k=1}^{\infty} h_k(\alpha_s) (\bar{\alpha}L)^k + O(L^0), \quad (2.4)$$

where

$$L \equiv \ln \frac{b_0^2}{Q^2 b^2} \quad (2.5)$$

is the large logarithm which is resummed, and  $O(L^0)$  denotes terms which are not logarithmically enhanced as  $b \rightarrow \infty$ . For future convenience, we have introduced in the definition of  $L$  an arbitrary constant  $b_0$  (to be discussed below), and we have further defined

$$\bar{\alpha} \equiv \beta_0 \alpha_s(Q^2), \quad (2.6)$$

$\beta_0$  is the first coefficient of the QCD beta function,

$$Q^2 \frac{\partial \alpha_s(Q^2)}{\partial Q^2} = -\beta_0 \alpha_s^2(Q^2) [1 + \beta_1 \alpha_s(Q^2) + O(\alpha_s^2)] \quad (2.7)$$

$$\beta_0 = \frac{33 - 2n_f}{12\pi}, \quad \beta_1 = \frac{1}{2\pi} \frac{153 - 19n_f}{33 - 2n_f}. \quad (2.8)$$

The inverse Fourier transform of  $\Sigma$  with respect to  $b$  is given by

$$\bar{\Sigma}(\alpha_s, \hat{q}_T^2) = \frac{Q^2}{2\pi} \int d^2b e^{-i\vec{q}_T \cdot \vec{b}} \Sigma(\alpha_s, \bar{\alpha}L) = \int_0^{+\infty} d\hat{b} \hat{b} J_0(\hat{b}\hat{q}_T) \Sigma(\alpha_s, \bar{\alpha}L), \quad (2.9)$$

using two-dimensional polar coordinates for  $\hat{b} \equiv bQ$ , and the integral representation of the 0-th order Bessel function,

$$J_0(z) = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-iz \cos \theta}. \quad (2.10)$$

Now consider specifically the resummation of

$$\bar{\Sigma}(\alpha_s, \hat{q}_T^2) = \frac{1}{\hat{\sigma}_0} \frac{d\hat{\sigma}}{d\hat{q}_T^2}, \quad (2.11)$$

where  $\frac{d\hat{\sigma}}{d\hat{q}_T^2}$  is the partonic transverse momentum distribution of a massive final state, and  $\hat{\sigma}_0$  the Born-level total cross-section. In this case, the  $b$ -space resummed result has the form [3]

$$\Sigma(\alpha_s, \bar{\alpha}L) = \exp S(\alpha_s, \bar{\alpha}L), \quad (2.12)$$

$$S(\alpha_s, \bar{\alpha}L) \equiv - \int_{\frac{b_0^2}{Q^2}}^{Q^2} \frac{d\mu^2}{\mu^2} \left[ \ln \frac{Q^2}{\mu^2} A(\alpha_s(\mu^2)) + B(\alpha_s(\mu^2)) \right], \quad (2.13)$$

where

$$A(\alpha_s) = A_1 \alpha_s + A_2 \alpha_s^2 + \dots; \quad B(\alpha_s) = B_1 \alpha_s + \dots, \quad (2.14)$$

and the constants  $A_i, B_i$  can be determined order by order by matching to the fixed-order calculation.

The integral in eq. (2.13) can be performed explicitly, and the result can then be expanded as

$$S(\alpha_s, \bar{\alpha}L) = \sum_{i=0}^{\infty} \bar{\alpha}^{i-1} f_i(\bar{\alpha}L), \quad (2.15)$$

where inclusion of the first  $k$  orders in the sum corresponds to the next<sup>k</sup>-to-leading log (N<sup>k</sup>LL) approximation. The LL and NLL functions  $f_0, f_1$  are explicitly given by

$$f_0(y) = \frac{A_1}{\beta_0} [\ln(1+y) - y] \quad (2.16)$$

$$f_1(y) = \frac{A_1\beta_1}{\beta_0^2} \left[ \frac{1}{2} \ln^2(1+y) - \frac{y}{1+y} + \frac{\ln(1+y)}{1+y} \right] - \frac{A_2}{\beta_0^2} \left[ \ln(1+y) - \frac{y}{1+y} \right] + \frac{B_1}{\beta_0} \ln(1+y). \quad (2.17)$$

Note that with  $y = \bar{\alpha}L$ , using the leading log form of  $\alpha_s(Q^2)$ ,

$$1+y = \frac{\alpha_s(Q^2)}{\alpha_s(b_0^2/b^2)}. \quad (2.18)$$

It is apparent from eqs. (2.16,2.17) that  $\Sigma(\alpha_s, \bar{\alpha}L)$  has a branch cut along the negative real axis in the complex plane of the variable  $y = \bar{\alpha}L$ :

$$\text{Re}(y) \leq -1; \quad \text{Im}(y) = 0. \quad (2.19)$$

This is due to the fact that the strong coupling blows up when its argument reaches the Landau pole, so that  $S(\alpha_s, \bar{\alpha}L)$  eq. (2.13) is singular when  $b$  becomes large enough, i.e. when

$$b^2 \geq b_L^2 \equiv \frac{b_0^2}{Q^2} e^{\frac{1}{\bar{\alpha}}}. \quad (2.20)$$

At leading order,  $b_L^2 = \frac{b_0^2}{\Lambda^2}$ . It follows that the series for  $\Sigma(\alpha_s, \bar{\alpha}L)$  eq. (2.4) has a finite radius of convergence, and the integrand in eq. (2.9) is not analytic in the whole integration range  $0 \leq \hat{b} < +\infty$ , so the Fourier inversion integral is not well-defined without a prescription to treat the singularity.

As mentioned in the introduction, various prescriptions of this kind have been proposed. Before discussing them, let us show that the reason why a prescription is needed is the divergence of the expansion in powers of  $\alpha_s(Q^2)$  of the resummed result obtained computing the inverse Fourier transform eq. (2.9) with  $\Sigma(\alpha_s, \bar{\alpha}L)$  eq. (2.12). To any finite perturbative order, the  $q_T$ -space resummed result is found by expanding eq. (2.12) and inverting the Fourier transform order by order:

$$\bar{\Sigma}_K(\alpha_s, \bar{L}) = \sum_{k=1}^K h_k(\alpha_s) \bar{\alpha}^k \frac{Q^2}{2\pi} \int d^2b e^{-i\vec{q}_T \cdot \vec{b}} L^k, \quad (2.21)$$

where we have replaced the argument  $\hat{q}_T^2$  of  $\bar{\Sigma}$  by

$$\bar{L} \equiv \ln \hat{q}_T^2 = \ln \frac{q_T^2}{Q^2}. \quad (2.22)$$

When  $K \rightarrow \infty$  the series eq. (2.21) diverges. To see this, we compute the integrals in eq. (2.21) using eq. (A.1) of the Appendix:

$$\bar{\Sigma}_K(\alpha_s, \bar{L}) = \frac{d}{d\hat{q}_T^2} R_K(\alpha_s, \bar{L}) \quad (2.23)$$

$$R_K(\alpha_s, \bar{L}) = 2 \sum_{k=1}^K h_k(\alpha_s) \bar{\alpha}^k \sum_{j=0}^k \binom{k}{j} M^{(j)}(0) \bar{L}^{k-j}, \quad (2.24)$$

where the function  $M(\eta)$  is defined in eq. (A.2), we have assumed  $\hat{q}_T^2 \neq 0$ , so that distributions can be ignored, and the term with  $j = k$ , which leads to a vanishing contribution to  $\bar{\Sigma}_K(\alpha_s, \bar{L})$ , has been included in the sum over  $j$  eq. (2.24) for later convenience. We now change the order of summation, and use the identity

$$\frac{1}{(k-j)!} = \frac{1}{2\pi i} \oint_H d\xi e^\xi \xi^{-(k-j)-1}, \quad (2.25)$$

where the integration path  $H$  is any closed contour which encloses the origin  $\xi = 0$ . We obtain

$$R_K(\alpha_s, \bar{L}) = 2 \sum_{j=0}^K \frac{M^{(j)}(0)}{j!} \sum_{k=j}^K \frac{k!}{(k-j)!} h_k \bar{\alpha}^k \bar{L}^{k-j} \quad (2.26)$$

$$= \frac{1}{\pi i} \oint_H \frac{d\xi}{\xi} e^\xi \sum_{j=0}^K \frac{M^{(j)}(0)}{j!} \left(\frac{\xi}{\bar{L}}\right)^j \sum_{k=j}^K k! h_k \left(\frac{\bar{\alpha}\bar{L}}{\xi}\right)^k. \quad (2.27)$$

Because of the singularity eq. (2.19), the power series eq. (2.4) has a finite radius of convergence equal to one

$$\lim_{k \rightarrow \infty} \left| \frac{h_{k+1}}{h_k} \right| = 1, \quad (2.28)$$

which immediately implies the vanishing of the radius of convergence of the sum over  $k$  in eq. (2.27).

The situation is thus similar to that which is encountered in threshold resummation [4, 8, 9]: the resummation is performed on quantities which are related by Mellin transformation to the physical ones, but the resummed results cannot be expressed as a Mellin transform of some function. Namely, their inverse Mellin transform does not exist, as a consequence of the fact that the inverse Mellin transform of their expansion in powers of  $\alpha_s(Q^2)$  diverges. In the present case, the divergence of the perturbative expansion implies that the Fourier inversion integral is ill-defined; of course the problem disappears if one retains only a finite number of terms in the resummed expansion [10, 11]. Various commonly used prescriptions replace the ill-defined integral with a well defined one, as we now review. In the next section, we construct a prescription which is instead based on the idea of replacing the divergent series with a convergent one through the Borel summation method. In the last section we will compare the various prescriptions and in particular the way they treat the divergence of the perturbative series.

In the prescription of ref. [3], the variable  $b$  is replaced by a function  $b_\star(b)$  which approaches a finite limit  $b_{\text{lim}} \leq b_L$  as  $b \rightarrow \infty$ , such as for example

$$b_\star = \frac{b}{\sqrt{1 + (b/b_{\text{lim}})^2}}. \quad (2.29)$$

In this way, the cut eq. (2.19) is never reached. This procedure has some degree of arbitrariness in the choice of the function  $b_\star(b)$ , which is interpreted as a parametrization of non-perturbative effects, whose size can be estimated by varying  $b_\star$ , for instance by changing the value of  $b_{\text{lim}}$ . The matching of this prescription to the fixed-order result is however numerically unstable, as pointed out in ref. [5].

A different possibility [5] is based on the observation that if only the leading log contribution (i.e. the terms with  $j = 0$ ) are included in eq. (2.24), then the series converges, and its sum can in fact be computed in closed form, with the result (see eq. (A.16) of the Appendix)

$$\bar{\Sigma}_{\text{LL}}(\alpha_s, \bar{L}) = 2 \frac{d}{d\hat{q}_T^2} \Sigma(\alpha_s, \bar{\alpha} \bar{L}). \quad (2.30)$$

Equation (2.13) implies that  $S(\alpha_s, \bar{\alpha} L)$  depends on  $b^2$  through  $\alpha_s(1/b^2)$ . Therefore, using eq. (2.18), the LL expression eq. (2.30) is seen to become a function of  $\alpha_s(q_T^2)$ . Therefore, the leading log truncation of the perturbative expansion in powers of  $\alpha_s(Q^2)$  eq. (2.27) has a finite radius of convergence, set by the Landau pole

$$q_T^2 > Q^2 \exp\left(-\frac{1}{\bar{\alpha}}\right) = \Lambda^2, \quad (2.31)$$

where the last equality holds at leading order.

The main defect of this result is that it is subject to large next-to-leading log corrections. In fact, the NLL Fourier inversion integral can also be computed in closed form [6]. The result (given in eq. (A.17)) differs sizably from the LL result even for relatively large values of  $q_T$  (several GeV for  $Q = 100$  GeV), as we shall see explicitly in Sect. 4 below. In fact, it turns out that the NLL correction diverges at a value of  $q_T$  which is an increasing function of the scale  $Q$ . This instability can be understood as a consequence of the fact that the truncation of the resummed result to finite logarithmic accuracy leads to an expansion in powers of  $\alpha_s(q_T^2)$  with coefficients depending on  $\ln(q_T/Q)$ , where higher powers of  $\alpha_s(q_T^2)$  correspond to higher logarithmic orders. Such an expansion is necessarily poorly behaved at low  $q_T$ , all the more so when the scale ratio  $q_T/Q$  is large. Performing the Fourier inversion to leading or next-to-leading logarithmic accuracy thus removes the divergence of the series eq. (2.21): this is analogous to what is found in the case of threshold resummation, where it can be shown [8] that the divergence of resummed results is removed if the Mellin inversion is performed to any finite logarithmic accuracy. However, the ensuing results are then perturbatively unstable.

A yet different way of treating the divergence has been proposed more recently in ref. [7], along the lines of the so-called Minimal Prescription of threshold resummation [4]. The basic idea here is that to any finite perturbative order, when the divergent series is replaced by a finite sum, one may choose the integration path in such a way that it avoids the singularities which appear at the

resummed level. The result of the Fourier (or respectively Mellin) inversion is then unchanged to any finite perturbative order, but it becomes finite at the resummed level. It can be further shown [4] that the divergent perturbative expansion of the resummed expression is asymptotic to the result obtained in this way. This prescription is widely used [2]: whereas in the case of threshold resummation it leads to dependence of resummed physical results on a kinematically inaccessible region (albeit by power-suppressed terms), in the case of transverse momentum resummation its only shortcoming is speed limitation in its numerical implementation.

### 3 The Borel prescription

We now turn to the construction of a prescription which extends to transverse momentum resummation the Borel prescription proposed in refs. [8, 9] for the resummation of threshold logarithms. The basic idea is to tackle directly the divergence of the series (2.24, 2.27) by summing it through the Borel method.

To do this, we take the Borel transform of eq. (2.27) with respect to  $\bar{\alpha}$ . This amounts to the replacement  $\bar{\alpha}^k \rightarrow w^{k-1}/(k-1)!$ , where  $w$  is the Borel variable conjugate to  $\bar{\alpha}$ . We obtain

$$\hat{R}_K(w, \bar{L}) = \frac{1}{\pi i} \oint_H \frac{d\xi}{\xi^2} e^{\bar{L}\xi} \sum_{j=0}^K \frac{M^{(j)}(0)}{j!} \xi^j \sum_{k=1}^K k h_k \left( \frac{w}{\xi} \right)^{k-1}, \quad (3.1)$$

where in comparison to eq. (2.27) we have rescaled the integration variable  $\xi \rightarrow \bar{L}\xi$ , and we have included all terms with  $1 \leq k \leq j-1$ , which vanish upon contour integration.

Both sums in eq. (3.1) are convergent as  $K \rightarrow \infty$ . Indeed,

$$\sum_{k=1}^{\infty} k h_k \left( \frac{w}{\xi} \right)^{k-1} = \xi \frac{d}{dw} \Sigma \left( \alpha_s, \frac{w}{\xi} \right) \quad \text{for } \left| \frac{w}{\xi} \right| < 1 \quad (3.2)$$

$$\sum_{j=0}^{\infty} \frac{M^{(j)}(0)}{j!} \xi^j = M(\xi) \quad \text{for } |\xi| < 1, \quad (3.3)$$

the last condition being due to the simple pole of  $M(\xi)$  at  $\xi = 1$ . Thus,

$$\hat{R}(w, \bar{L}) = \lim_{K \rightarrow \infty} \hat{R}_K(w, \bar{L}) = \frac{1}{\pi i} \oint_H \frac{d\xi}{\xi} e^{\bar{L}\xi} M(\xi) \frac{d}{dw} \Sigma \left( \alpha_s, \frac{w}{\xi} \right), \quad (3.4)$$

provided the contour  $H$  is chosen so that

$$w < |\xi| < 1. \quad (3.5)$$

Since  $M(\xi)$  has no singularities on the negative real axis, and  $\Sigma(\alpha_s, w/\xi)$  has a branch cut on the real  $\xi$  axis between  $-w$  and 1, the integration contour can now be deformed so that  $\hat{R}(w, \bar{L})$  is well defined for all positive values of  $w$  (see fig. 1).

The original function eq. (2.24) is recovered by inverting the Borel transform:

$$R(\alpha_s, \bar{L}) = \int_0^\infty dw e^{-\frac{w}{\bar{\alpha}}} \hat{R}(w, \bar{L}). \quad (3.6)$$



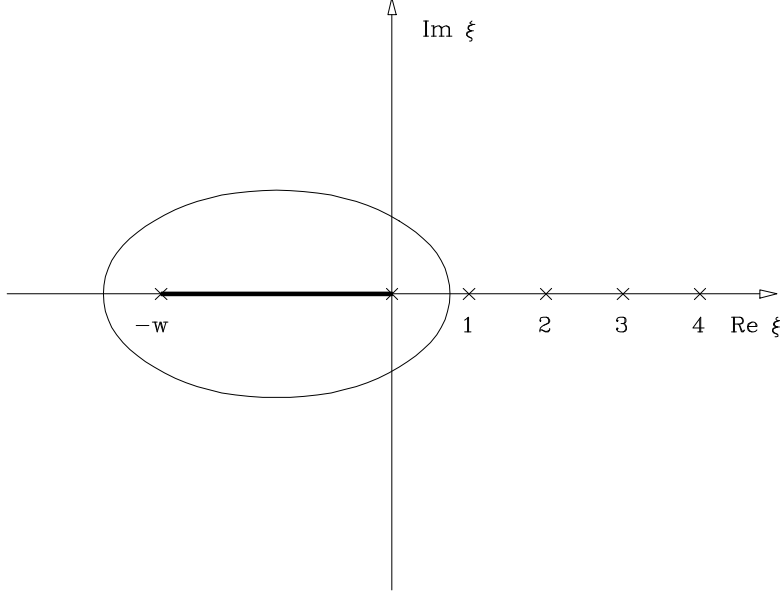


Figure 1: The integration contour  $H$  in eq. (3.4).

The inversion integral is divergent at  $w \rightarrow \infty$ . This is easily seen by inspection of fig. 1: as  $w$  becomes large, the branch cut extends to the left, and the integration contour is pushed towards large negative values of  $\xi$ , where  $M(\xi)$  oscillates with a factorially growing amplitude.

We regulate the integral by cutting it off at  $w = C$ . We thus get

$$R^C(\alpha_s, \bar{L}) = \frac{1}{\pi i} \oint_H \frac{d\xi}{\xi} M(\xi) e^{\bar{L}\xi} \int_0^C dw e^{-\frac{w}{\bar{\alpha}}} \frac{d}{dw} \Sigma \left( \alpha_s, \frac{w}{\xi} \right), \quad (3.7)$$

which is the Borel prescription for transverse momentum resummation. The result can be equivalently rewritten by doing a partial integration as

$$R^C(\alpha_s, \bar{L}) = \frac{1}{\pi i} \oint_H \frac{d\xi}{\xi} M(\xi) e^{\bar{L}\xi} \left[ e^{-\frac{C}{\bar{\alpha}}} \Sigma \left( \alpha_s, \frac{C}{\xi} \right) + \frac{1}{\bar{\alpha}} \int_0^C dw e^{-\frac{w}{\bar{\alpha}}} \Sigma \left( \alpha_s, \frac{w}{\xi} \right) \right], \quad (3.8)$$

which may be more convenient for numerical implementations in that it depends directly on the physical observable  $\Sigma$ , rather than its derivative. Equation (3.8), and its equivalent form eq. (3.7), are the main result of this paper. It is interesting to observe that if we integrate by parts before cutting off the integral, then the surface term vanish. We then end up with the alternative resummation

$$R^{C'}(\alpha_s, \bar{L}) = \frac{1}{\pi i} \oint_H \frac{d\xi}{\xi} M(\xi) e^{\bar{L}\xi} \frac{1}{\bar{\alpha}} \int_0^C dw e^{-\frac{w}{\bar{\alpha}}} \Sigma \left( \alpha_s, \frac{w}{\xi} \right). \quad (3.9)$$

As we shall see shortly, this is an equally valid prescription.

In order to see that this is a valid resummation prescription, consider the truncation to order

$K$  of eq. (3.6), namely

$$\begin{aligned} R_K^C(\alpha_s, \bar{L}) &\equiv \int_0^C dw e^{-\frac{w}{\bar{\alpha}}} \hat{R}_K(w, \bar{L}) \\ &= 2 \sum_{j=0}^K \frac{M^{(j)}(0)}{j!} \sum_{k=j}^K \frac{k!}{(k-j)!} h_k \bar{\alpha}^k \frac{\gamma(k, \frac{C}{\bar{\alpha}})}{(k-1)!} \bar{L}^{k-j}, \end{aligned} \quad (3.10)$$

where

$$\gamma(k, z) = \int_0^z dw e^{-w} w^{k-1} = (k-1)! \left( 1 - e^{-z} \sum_{n=0}^{k-1} \frac{z^n}{n!} \right) \quad (3.11)$$

is the truncated gamma function. The difference between the original  $R_K(\alpha_s, \bar{L})$  eq. (2.23) and its Borel resummation  $R_K^C(\alpha_s, \bar{L})$  is

$$\begin{aligned} R_K^{\text{ht}}(\alpha_s, \bar{L}; C) &\equiv R_K(\alpha_s, \bar{L}) - R_K^C(\alpha_s, \bar{L}) \\ &= 2 e^{-\frac{C}{\bar{\alpha}}} \sum_{j=0}^K \frac{M^{(j)}(0)}{j!} \sum_{k=j}^K \frac{k!}{(k-j)!} h_k \bar{\alpha}^k \bar{L}^{k-j} \sum_{n=0}^{k-1} \frac{1}{n!} \left( \frac{C}{\bar{\alpha}} \right)^n. \end{aligned} \quad (3.12)$$

Because

$$e^{-\frac{C}{\bar{\alpha}}} = \left( \frac{\Lambda^2}{Q^2} \right)^C [1 + O(\alpha_s(Q^2))], \quad (3.13)$$

$R_K^{\text{ht}}(\alpha_s, \bar{L}; C)$  is seen to be power-suppressed at large  $Q^2$  (higher twist): cutting off the  $w$  integration at  $w = C$  is equivalent to the inclusion of a higher twist term, which cancels the divergence of the resummed expression. Specifically,  $R_K^{\text{ht}}(\alpha_s, \bar{L}; C)$  is a twist- $t$  contribution with

$$t = 2(1 + C), \quad (3.14)$$

the choice  $C = 1$  corresponds to the inclusion of a twist-four term. Moreover, it is apparent from eq. (3.12) that

$$R_K^{\text{ht}}(\alpha_s, \bar{L}; C) \underset{\alpha_s \rightarrow 0}{\sim} e^{-\frac{C}{\bar{\alpha}}}, \quad (3.15)$$

which vanishes faster than any power of  $\alpha_s$  as  $\alpha_s \rightarrow 0$ . It follows that the original divergent  $R_K(\alpha_s, \bar{L})$  is an asymptotic expansion of the Borel-resummed result  $R^C(\alpha_s, \bar{L})$  eqs. (3.8, 3.7).

Furthermore, the alternative prescription  $R^{C'}(\alpha_s, \bar{L})$  eq. (3.9) differs from  $R^C(\alpha_s, \bar{L})$  eq. (3.8) by the first term in square brackets in (3.8), which is a finite higher-twist contribution. Hence, the two prescriptions correspond to two inequivalent but equally acceptable regularizations of the divergent sum which differ by finite terms, and are both asymptotic sums of the divergent series.

The main features of the Borel prescription can be appreciated by considering as an explicit example of a resummed quantity  $\Sigma(\alpha_s, \bar{\alpha}L) = \gamma_{\text{LL}}(\alpha_s, \bar{\alpha}L)$ , with

$$\gamma_{\text{LL}}(\alpha_s, \bar{\alpha}L) \equiv \frac{dS_{\text{LL}}(\alpha_s, \bar{\alpha}L)}{d \ln Q^2}, \quad (3.16)$$

and  $S_{\text{LL}}(\alpha_s, \bar{\alpha}L)$  given by eqs. (2.13,2.15) evaluated at the leading log level (2.16), namely

$$\gamma_{\text{LL}}(\alpha_s, \bar{\alpha}L) = \frac{A_1}{\beta_0} \ln(1 + \bar{\alpha}L). \quad (3.17)$$

Substituting this form of  $\Sigma(\alpha_s, \bar{\alpha}L)$  in eq. (3.7), the associate  $q_T$ -space physical observable computed with the Borel prescription is found to be

$$\bar{\gamma}_{\text{LL}}^C(\alpha_s, \bar{L}) = \frac{A_1}{\beta_0} \frac{1}{\hat{q}_T^2} \int_0^C dw e^{-\frac{w}{\bar{\alpha}}} \frac{1}{\pi i} \oint_H d\xi M(\xi) e^{\bar{L}\xi} \frac{1}{\xi + w}. \quad (3.18)$$

The  $\xi$  integral is easy to calculate, because the integrand has only a simple pole at  $\xi = -w$ :

$$\bar{\gamma}_{\text{LL}}^C(\alpha_s, \bar{L}) = \frac{2A_1}{\beta_0} \frac{1}{\hat{q}_T^2} \int_0^C dw \left( \frac{\Lambda^2}{q_T^2} \right)^w M(-w), \quad (3.19)$$

where we have used the leading-log expression of the running coupling. It is thus clear that the divergent integration is cut off by the inclusion of a power-suppressed contribution

$$\begin{aligned} \bar{\gamma}_{\text{LL}}^{\text{ht}}(\alpha_s, \bar{L}; C) &= \frac{2A_1}{\beta_0} \frac{1}{\hat{q}_T^2} \int_C^{+\infty} dw \left( \frac{\Lambda^2}{q_T^2} \right)^w M(-w) \\ &= \frac{2A_1}{\beta_0} \frac{1}{\hat{q}_T^2} \left( \frac{\Lambda^2}{q_T^2} \right)^C \int_0^{+\infty} dw \left( \frac{\Lambda^2}{q_T^2} \right)^w M(-w - C). \end{aligned} \quad (3.20)$$

Note that the suppression is by powers of  $\frac{\Lambda^2}{q_T^2}$ : at finite order  $K$  the higher twist contribution is suppressed by a power of  $\frac{\Lambda^2}{Q^2}$ , as shown in eq. (3.12), but when resummed to all orders, the scale  $Q^2$  is replaced by an effective scale  $q_T^2$ .

## 4 Comparison of resummation prescriptions

Let us now compare the results found using the Borel prescription to those of other prescriptions, with the dual goal of understanding the advantages and disadvantages of various methods, and of assessing the ambiguity which is intrinsic to the resummation of a divergent expansion.

First, we look at a typical resummed observable. Namely, we consider the transverse momentum distribution of Drell-Yan pairs, eq. (2.11), which we evaluate at the partonic resummed next-to-leading log level, i.e. using eq. (2.12) with  $S(\alpha_s, \bar{\alpha}L)$  computed including the first two terms in eq. (2.15), given in eqs. (2.16,2.17) with [12, 13]

$$A_1 = \frac{C_F}{\pi} \quad (4.1)$$

$$A_2 = \frac{1}{\pi^2} \left( \frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{27} n_f + \frac{8\pi}{3} \beta_0 \ln \frac{b_0 e^{\gamma_E}}{2} \right) \quad (4.2)$$

$$B_1 = \frac{2C_F}{\pi} \ln \frac{b_0 e^{\gamma_E - 3/4}}{2}. \quad (4.3)$$

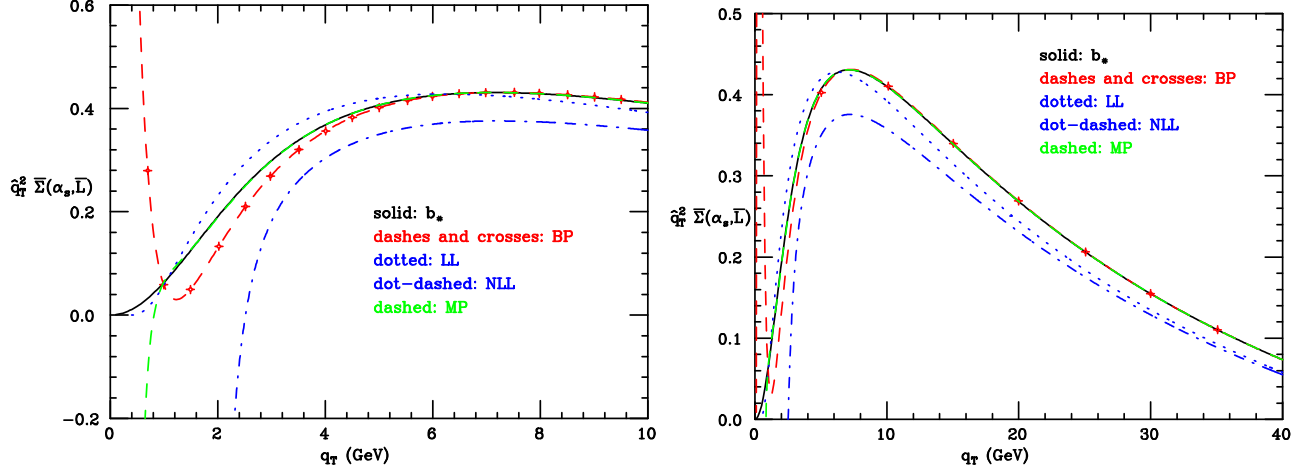


Figure 2: The NLL partonic resummed Drell-Yan transverse momentum distribution computed with various resummation prescriptions with  $Q^2 = 10^4 \text{ GeV}^2$  and in a narrow (left) and wide (right) range of  $q_T$ .

The results are displayed in fig. 2, for  $Q = 100 \text{ GeV}$ . The two lower curves at large  $q_T$  in this figure correspond to those found using respectively eqs. (A.16) and eqs. (A.17) of the Appendix, namely, to inverting the Fourier transform to leading and next-to-leading log accuracy (with  $b_0 = 2e^{-\gamma_E}$ ). The sizable difference between these two results even for  $q_T$  as large as 10 GeV shows the instability of the truncation of the Fourier transform to finite log accuracy discussed in the introduction and first stressed in ref. [6].

The other prescriptions displayed in fig. 2 are the  $b_*$  prescription, where the Fourier inversion is performed after replacing  $b$  with  $b_*$  eq. (2.29), with  $b_{\text{lim}} = b_L$ , where  $b_L = 7.2 \text{ GeV}^{-1}$  is the NLO Landau pole eq. (2.20); the minimal prescription (MP) where the Fourier inversion is performed along the deformed path of ref. [7], and the Borel prescription eq. (3.7) with  $C = 1$ .

In fig. 3 we further show the dependence of the Borel prescription on the parameter  $C$  which characterizes the higher twist term included in the resummation eqs. (3.13,3.14), as it is varied between twist four and twist eight. Because all these choices provide valid resummation prescriptions, this variation provides an estimate of the ambiguity which is intrinsic of the resummation procedure: indeed, the  $b_*$  and minimal prescription, also shown in this figure, are well within the band of variation as  $q_T \rightarrow 0$ . These plots show that the ambiguity in the resummation procedure is negligible for  $q_T \gtrsim 5 \text{ GeV}$ , it remains small for  $q_T \gtrsim 2 \text{ GeV}$ , and it only blows up as  $q_T$  approaches the Landau pole.

We can further elucidate the origin of these results by studying the effect of the various prescriptions when the divergent sum eq. (2.21) is truncated, so the Fourier inversion can be performed term by term. Consider specifically the first term in the series, namely, the inverse Fourier transform of  $L$ . The exact result is given by eq. (A.1) for  $k = 1$ ,

$$\frac{1}{2\pi} \int d^2\hat{b} e^{-i\hat{q}_T \cdot \hat{b}} \ln \frac{b_0^2}{\hat{b}^2} = \frac{2}{\hat{q}_T^2}. \quad (4.4)$$

The MP reproduces this exact result, because  $\ln(b_0^2/\hat{b}^2)$  is analytic on the positive real  $\hat{b}$  axis,

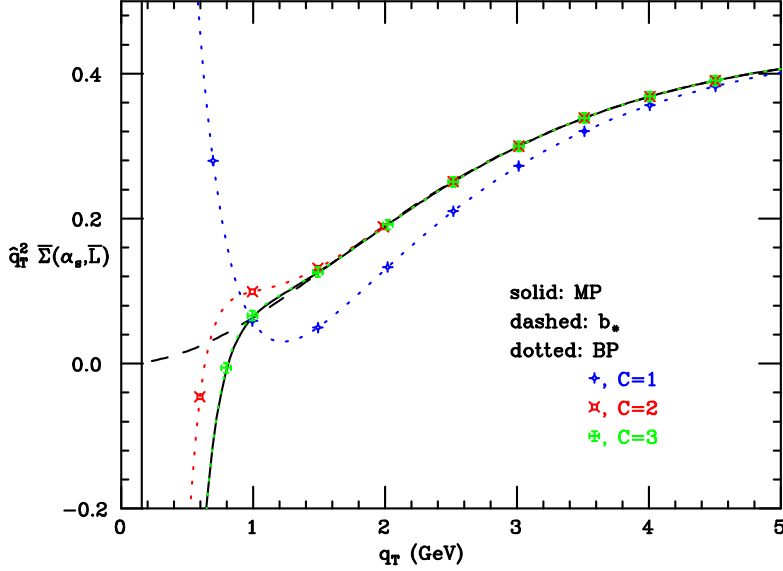


Figure 3: Dependence of the results shown in fig. 2 for the Borel prescription on the parameter  $C$ . The vertical line at  $q_T = 156$  MeV indicates the position of the Landau pole.

and a deformation of the integration contour has no effect; a branch cut on the positive real  $\hat{b}$  axis only arises after summation of the whole series.

The Borel prescription yields instead

$$\left[ \frac{1}{2\pi} \int d^2 \hat{b} e^{-i\hat{q}_T \cdot \hat{b}} \ln \frac{b_0^2}{\hat{b}^2} \right]_{\text{BP}} = \frac{2}{\hat{q}_T^2} \left( 1 - e^{-\frac{C}{\bar{\alpha}}} \right) \quad (4.5)$$

as one can see by setting  $h_1 = \frac{1}{\bar{\alpha}}$  and  $h_k = 0$  for all  $k \neq 1$  in eq. (3.10). The exact result is modified by the introduction of a correction of twist  $2(1+C)$ . Note that the higher twist correction is tiny at large  $Q^2$ , of order  $10^{-6}$  for  $C = 1$  and  $Q^2 = 10^4$  GeV<sup>2</sup>. If we use the alternative Borel prescription  $R^{C'}(\alpha_s, \bar{L})$  eq. (3.9) we get instead

$$\left[ \frac{1}{2\pi} \int d^2 \hat{b} e^{-i\hat{q}_T \cdot \hat{b}} \ln \frac{b_0^2}{\hat{b}^2} \right]_{\text{BP}'} = \frac{2}{\hat{q}_T^2} \left[ 1 - e^{-\frac{C}{\bar{\alpha}}} \left( 1 + \frac{C}{\bar{\alpha}} \right) \right]. \quad (4.6)$$

so the two prescriptions are indeed seen to differ by a higher twist term.

Finally, the result of the replacement of  $b$  by  $b_*$  eq. (2.29) can be computed analytically in terms of the Bessel function  $K_1$ :

$$\begin{aligned} \frac{1}{2\pi} \int d^2 \hat{b} e^{-i\hat{q}_T \cdot \hat{b}} \ln \frac{b_0^2}{\hat{b}_*^2} &= \frac{1}{2\pi} \int d^2 \hat{b} e^{-i\hat{q}_T \cdot \hat{b}} \ln \left[ \frac{b_0^2}{\hat{b}^2} \left( 1 + \frac{\hat{b}^2}{\hat{b}_{\text{lim}}^2} \right) \right] \\ &= \frac{2}{\hat{q}_T^2} \left[ 1 - \hat{b}_{\text{lim}} \hat{q}_T K_1(\hat{b}_{\text{lim}} \hat{q}_T) \right]. \end{aligned} \quad (4.7)$$

Using the asymptotic behaviour  $K_1(z) \underset{z \rightarrow \infty}{\sim} e^{-z}/\sqrt{z}$ , we see that the correction factor in eq. (4.7) vanishes faster than any power of  $1/(b_{\text{lim}} q_T)$  for  $q_T \gg 1/b_{\text{lim}}$ .

For higher order powers of  $L$  the same qualitative behaviour is found using the various prescriptions discussed here. Namely, the MP gives the exact Fourier transform eq. (A.1); the BP gives a result which differs from it by a higher twist term, and the  $b_\star$  prescription gives a result which differs from it by a term which is exponentially suppressed in  $1/(b_{\text{lim}}q_T)$ .

We thus see that the way different prescriptions tackle the divergence of the perturbative expansion is the following. In the LL and NLL case, the divergent series eq. (2.24) is made convergent by truncating the Fourier inversion to finite order, i.e. by only retaining a finite number of terms in the inner sum over  $j$ . This, as discussed in Section 2, leads effectively to an expansion in powers of  $\alpha_s(q_T^2)$  which has very poor convergence properties at small  $q_T$  even when  $Q$  is large. The MP and BP both provide an asymptotic sum of the divergent series: the BP removes the divergence by inclusion of a higher twist term, and the MP by a suitable analytic continuation, which corresponds [4] to the inclusion of terms which are more suppressed than any power of  $Q^2$ . At large  $Q^2$ , the higher twist term of the BP is negligible so these two prescriptions are essentially indistinguishable when applied to convergent series. When applied to the divergent resummed expansion displayed in figs. 2-3 they only differ in the region where  $q_T$  approaches the Landau pole, so the high-order behaviour of the series become relevant. Finally, the  $b_\star$  prescription modifies the divergent series by inclusion of a term which is more suppressed than any power of  $1/(b_{\text{lim}}q_T)$ . When applied to a convergent series, this prescription produces a result that differs sizably from that of the BP when  $q_T^2 \ll Q^2$  and it approaches the Landau pole: this is because the scale of the correction term is set by  $Q^2$  for the BP, and by  $q_T^2$  for the  $b_\star$  prescription. At the resummed level, however, the effective scale of power suppressed terms becomes  $q_T^2$  also for the BP (compare eq. (3.20)), so all resummation prescriptions lead essentially to the same result.

## 5 Summary

We have constructed a resummation prescription for transverse momentum distributions which extends to this case the Borel prescription previously proposed for threshold resummation [8, 9]. The construction is based on the observation that the reason why a resummation prescription is needed in the first place is that the perturbative expansion of resummed results in  $q_T$  space in powers of  $\alpha_s(Q^2)$  diverges. The Borel prescription tackles this divergence by summing the convergent Borel transform of the divergent series, and then making the Borel inversion finite by inclusion of a higher twist term. The original divergent series is an asymptotic expansion of the result obtained thus. The Borel prescription is easily amenable to numerical implementation; being based on a  $b$ -space resummation it is easy to match to fixed-order results, and it is perturbatively stable.

There is some freedom in this prescription, parametrized by a real parameter  $C$ , related to the twist  $t$  of the term included in order to obtain convergence by  $t = 2(C + 1)$ . Whereas  $C$  may be chosen to take any value, it is convenient to choose a value which corresponds to twists which already appear in the expansion of the observable being considered. Indeed, physical observables must be independent of the choice of  $C$ , and thus if an unphysical twist term is introduced, it must be compensated by an equal and opposite power suppressed term which is

thereby artificially introduced by this choice.

Comparison of the Borel prescription to other available resummations, such as the minimal prescription or the  $b_*$  method, shows that at large  $Q^2$  they lead to results which are extremely stable and which only differ when  $q_T$  approaches the Landau pole. In fact, variation of the parameter  $C$  of the Borel prescription provides a reliable estimate of the ambiguity in the resummation procedure. For  $q_T \gtrsim 2$  GeV this ambiguity appears to be negligibly small, even in the region of a few GeV where the impact of the resummation is sizable. This is in contrast to the case of threshold resummation, where it was found [9] that the ambiguity is almost as large as the effect of the resummation itself in most of the kinematic region where the resummation is relevant.

Our results contradict the widespread prejudice that transverse momentum resummation is affected by sizable ambiguities, and it shows that, at least as long as  $Q$  is as large as the  $W$  mass and  $q_T$  as large as the nucleon mass perturbative resummation of transverse momentum distributions provides reliable and stable results. The Borel prescription provides a new method for performing this resummation which has more stable matching properties than the  $b_*$  prescription and might be numerically advantageous over the widely used minimal prescription.

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## A Appendix

In this appendix, we collect some results on two-dimensional Fourier transforms of powers of logarithms.

First, we compute the exact Fourier transform with respect to  $\hat{b}$  of the  $k$ -th power of  $\ln^k \frac{b_0^2}{\hat{b}^2}$  (with  $b_0$  a constant). We get

$$\frac{1}{2\pi} \int d^2 \hat{b} e^{-i \hat{q}_T \cdot \hat{b}} \ln^k \frac{b_0^2}{\hat{b}^2} = 2 M^{(k)}(0) \delta(\hat{q}_T^2) + 2 \sum_{j=0}^{k-1} \binom{k}{j} M^{(j)}(0) \left[ \frac{d}{d\hat{q}_T^2} \ln^{k-j} \hat{q}_T^2 \right]_+, \quad (\text{A.1})$$

where

$$M(\eta) = \left( \frac{b_0^2}{4} \right)^\eta \frac{\Gamma(1-\eta)}{\Gamma(1+\eta)}, \quad (\text{A.2})$$

and the  $+$  distributions are defined by

$$\int_0^1 d\hat{q}_T^2 [D(\hat{q}_T^2)]_+ = 0. \quad (\text{A.3})$$

In order to prove eq. (A.1), we define a generating function

$$\chi(\hat{b}, \eta) = \left( \frac{b_0^2}{\hat{b}^2} \right)^\eta; \quad L^k = \ln^k \frac{b_0^2}{\hat{b}^2} = \frac{\partial^k}{\partial \eta^k} \chi(\hat{b}, \eta) \Big|_{\eta=0}. \quad (\text{A.4})$$

We have

$$\frac{1}{2\pi} \int d^2 \hat{b} e^{-i \hat{q}_T \cdot \hat{b}} \chi(\hat{b}, \eta) = \int_0^{+\infty} d\hat{b} \hat{b} J_0(\hat{b} \hat{q}_T) \left( \frac{b_0^2}{\hat{b}^2} \right)^\eta, \quad (\text{A.5})$$

where we have used polar coordinates for  $\hat{b}$ , and the integral representation of the 0-th order Bessel function

$$J_0(z) = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-iz \cos \theta}. \quad (\text{A.6})$$

The integral can be computed by means of the identity

$$\int_0^{+\infty} dx x^\mu J_\nu(ax) = 2^\mu a^{-\mu-1} \frac{\Gamma(\frac{1}{2} + \frac{\nu}{2} + \frac{\mu}{2})}{\Gamma(\frac{1}{2} + \frac{\nu}{2} - \frac{\mu}{2})} \quad a > 0; \quad -\text{Re } \nu - 1 < \text{Re } \mu < \frac{1}{2}. \quad (\text{A.7})$$

We find

$$\frac{1}{2\pi} \int d^2 \hat{b} e^{-i \hat{q}_T \cdot \hat{b}} \chi(\hat{b}, \eta) = 2 \eta M(\eta) (\hat{q}_T^2)^{\eta-1}. \quad (\text{A.8})$$

We may now replace

$$(\hat{q}_T^2)^{\eta-1} = \left[ (\hat{q}_T^2)^{\eta-1} \right]_+ + \frac{1}{\eta} \delta(\hat{q}_T^2), \quad (\text{A.9})$$

consistent with the definition eq. (A.3). We get

$$\frac{1}{2\pi} \int d^2 \hat{b} e^{-i \hat{q}_T \cdot \hat{b}} \chi(\hat{b}, \eta) = 2 M(\eta) \left\{ \delta(\hat{q}_T^2) + \left[ \frac{d}{d\hat{q}_T^2} (\hat{q}_T^2)^\eta \right]_+ \right\}. \quad (\text{A.10})$$



Evaluating the  $k$ -th derivative of both sides with respect to  $\eta$  at  $\eta = 0$  leads immediately to eq. (A.1). Note that the term  $j = k$  is excluded from the sum because it vanishes upon differentiation with respect to  $\hat{q}_T^2$ . For  $\hat{q}_T^2$  strictly larger than zero, both the term proportional to  $\delta(\hat{q}_T^2)$  and the  $+$  prescription have no effect.

Let us now turn to the evaluation of the Fourier transform to fixed logarithmic accuracy. Equation (A.1) shows that the Fourier transform of the  $k$ -th power of  $\ln b$  is proportional to  $1/\hat{q}_T^2$  times the  $(k - 1)$ -th power of the log of the Fourier conjugate variable  $\ln \hat{q}_T^2$  (leading log approximation), but also includes terms proportional to all lower powers of this log. The  $N^{\text{LL}}$  approximation corresponds to including terms up to  $j = n$  in the sum in eq. (A.1), i.e. such that the power of  $\ln \hat{q}_T^2$  is by  $n + 1$  units lower than the power of  $\ln(b_0^2/\hat{b}^2)$ .

The NLL and  $N^2\text{LL}$  approximations are particularly simple due to the fact that

$$M^{(1)}(0) = \ln \frac{b_0^2}{4} + 2\gamma_E \quad (\text{A.11})$$

$$M^{(2)}(0) = \left( \ln \frac{b_0^2}{4} + 2\gamma_E \right)^2 \quad (\text{A.12})$$

where  $\gamma_E \approx 0.5772$  is the Euler constant. It follows in particular that if  $b_0 = 2e^{-\gamma_E}$ , the NLL and NNLL terms in eq. (A.1) vanish [12].

A useful form of the  $N^{\text{LL}}$  approximation can be obtained noting that

$$M^{(j)}(0) = \int_0^\infty dx J_1(x) \ln^j \frac{b_0^2}{x^2}. \quad (\text{A.13})$$

It follows that eq. (A.1) (for  $\hat{q}_T^2 > 0$ , i.e. neglecting distributions) can be rewritten as

$$\frac{1}{2\pi} \int d^2\hat{b} e^{-i\hat{q}_T \cdot \hat{b}} \ln^k \frac{b_0^2}{\hat{b}^2} = 2 \frac{d}{d\hat{q}_T^2} \int_0^\infty dx J_1(x) \left( \ln \hat{q}_T^2 + \ln \frac{b_0^2}{x^2} \right)^k. \quad (\text{A.14})$$

The  $N^{\text{LL}}$  approximation can then be obtained by retaining the first  $n$  terms in the binomial expansion of  $\left( \ln \hat{q}_T^2 + \ln \frac{b_0^2}{x^2} \right)^k$  in this equation.

This result is particularly useful in that it allows the computation in closed form of some Fourier transforms of generic functions to fixed logarithmic accuracy. Specifically, consider a function

$$F(L) = \sum_{k=0}^\infty F_k L^k. \quad (\text{A.15})$$

Its Fourier transform to LL accuracy is given by

$$\left[ \frac{1}{2\pi} \int d^2\hat{b} e^{-i\hat{q}_T \cdot \hat{b}} F(L) \right]_{\text{LL}} = 2 \frac{d}{d\hat{q}_T^2} \sum_{k=0}^\infty F_k \int_0^\infty dx J_1(x) \ln^k \hat{q}_T^2 = 2 \frac{d}{d\hat{q}_T^2} F(\ln \hat{q}_T^2). \quad (\text{A.16})$$

This result was given in ref. [5].

One may think that because of eqs. (A.11-A.12) eq. (A.16) with  $b_0 = 2e^{-\gamma_E}$  automatically provides a result which is correct to  $N^2\text{LL}$  accuracy. This, however, is not true if  $F(L)$  is

a physical observable, such as a cross-section. Indeed, in this case the N<sup>n</sup>LL approximation to it is defined by expansion of its logarithm: for example if  $F(L)$  is identified with  $\Sigma(\alpha_s, \bar{\alpha} L)$  eq. (2.12), the expansion of it to subsequent logarithmic order is given by the expansion eq. (2.15) of  $S(\alpha_s, \bar{\alpha} L) = \ln \Sigma(\alpha_s, \bar{\alpha} L)$ , and not of  $\Sigma(\alpha_s, \bar{\alpha} L)$  itself. The NLL approximation to the Fourier inverse of  $F(L)$  may however be calculated exactly in terms of  $G(L) \equiv \ln F(L)$ . One finds

$$\begin{aligned} \left[ \frac{1}{2\pi} \int d^2 \hat{b} e^{-i \hat{q}_T \cdot \hat{b}} F(L) \right]_{\text{NLL}} &= 2 \frac{d}{d \hat{q}_T^2} \int_0^\infty dx J_1(x) \exp \left[ G_0 + G_1 \ln \frac{b_0^2}{x^2} \right] \\ &= 2 \frac{d}{d \hat{q}_T^2} F(\ln \hat{q}_T^2) M \left( G'(\ln \hat{q}_T^2) \right), \end{aligned} \quad (\text{A.17})$$

where

$$G(L) \equiv \ln F(L) = G_0 + G_1 \ln \frac{b_0^2}{x^2} + O \left( \ln^2 \frac{b_0^2}{x^2} \right), \quad (\text{A.18})$$

with

$$G_0 = G(\ln \hat{q}_T^2); \quad G_1 = G'(\ln \hat{q}_T^2). \quad (\text{A.19})$$

This is the result found in ref. [6].

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